

On the Study of Richard Tom Robert Identity

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Abstract

The evolution this article is dramatic. In order to estimate the average speed of mosquitoes, a simple experiment was designed by Richard (Lu-Hsing Tsai), Tom (Po-Yu Tsai) and Robert (Hung-Ming Tsai). The result of the experiment was posted in the science exhibitions Taichung Taiwan 1993. The average speed of mosquitoes is inferred by the simple relation $m = K\rho cAt$, where m is the number of mosquitoes that are caught by the trap, ρ is the density of the mosquitoes in the experimental device (a camp tent), c is the mean speed of the mosquitoes, A is the effect area of the trap, t is the time interval, and K is a proportional constant. In order to find the value of K . The identity $E(l) = \alpha V/A$ is discovered, where V is the volume of a bounded region in the space R^n , A is the boundary area of the region, α is a proportional constant, $E(l)$ is the expectation of the random paths. By a random path in the region we mean the path which is traced by a random walker who passes through the region. In this paper, we will show how to get the data of α generated by computer. Actually, $\alpha = 1/K$. Though the rigorous proof is not shown, a sketch proof will be shown in this paper. The theoretical values of K are $1/2$, $1/\pi$, $1/4$, $2/(3\pi)$, ..., for dimension $n = 1, 2, 3, 4, \dots$,

Introduction

Although random walk is just a simple mathematical model, it can be used to describe many physical phenomena such as Brownian motion and diffusion. There is a very important number called "Boltzmann constant" in physics. Historically, scientists used the random walk model to evaluate this constant for the first time [4]. The Bertrand paradox can be traced back to the Buffon needle problem established in 1777 [1]. Since then, the research of Buffon needle problem has been developed into a branch of mathematics, called geometric probability. The most important subject in geometric probability is "random chords" [1]. It is very difficult to design an experiment based on its original definition because an infinite long straight line is required. To the best of our knowledge, no one has mentioned the relationship between the random walk model and the random chord problem. In this article, we use a simple way to show that random walk problem is essentially a random chords problem.

A simple experiment

In 1993, Hung-Ming Tsai [6] had an idea to estimate the mean speed of mosquitoes. The idea was: mosquitoes were put in a hemispherical tent with a diameter of 230 cm, and a trap was set up somewhere in the tent. When a mosquito touched the trap, it would get killed because there we electrified grids on the trap. And a electric spark made a sound which was sensed by human ears. After the mosquitoes were put in the tent uniformly, the switch of

the trap was turned on, the number of sparks was counted, and the time was recorded. The idea was nothing other than an experiment. By this experiment, he estimated the mean speed of mosquitoes.

What described above is a crude but interesting experiment. Although the experiment is easy, the mathematics for analyzing the data obtained from the experiment seems not trivial. Let t be the time interval in which there are m mosquitoes that are trapped. Let ρ be the density of mosquitoes in the tent, let c be the mean speed of the mosquitoes and let A be the effective area of the trap. In order to analyze the experimental data, we assume

$$m = K\rho cAt \quad (1)$$

where K is a proportional constant. From the experiment, we can easily get data for m , ρ , A , and t . If we assume that the flying of mosquitoes is a three dimensional random walk, by applying the theory of geometric probability to compute the value of K , then the mean speed of mosquitoes can be estimated from equation (1). We find that it is nontrivial to find the theoretical values of K . Due to the Bertrand's paradox and the difficulty, we compute the value of K by computer simulation

The value of K and a new mathematical identity

Let $E(l)$ be the expectation of the length of random paths in the bounded region. Without loss the generality, we assume the bounded region is a sphere. In order to simplify the sketch proof of the identity, we use discrete model. If it is necessary to treat the continuous case, then the radius of sphere is taken by limit process. Let $\tau = E(l)/c$, where c is the mean speed of mosquitoes. Then τ is the mean time interval that a mosquito stays in the sphere. Consider the sequence of time $t_0, t_1, \dots, t_i, \dots$, where $t_i = i \cdot \tau$, $i = 0, 1, 2, \dots$. Imagine that at each time t_i , there are exactly N (new) mosquitoes get into the sphere and N (old) mosquitoes get out of the sphere. So, at any time, there are exactly N mosquitoes in the sphere. By the definition of density, we have $\rho = N/V$, where V is the volume of the sphere. Now, we consider that the surface of the sphere is the sensor whose area is A . During any time interval of length τ , there are exactly N mosquitoes passing through the boundary and getting into the sphere. So the boundary (or the sensor) senses N mosquitoes, during a time interval of τ , that get into the sphere and hence senses Nt/τ mosquitoes in any time interval of t . Since Nt/τ is the m in equation (1), $\rho = N/V$, and $\tau = E(l)/c$, equation (1) becomes

$$K = A/(VE(l)) \quad (2)$$

The mathematical identity is obtained so easily that we can not almost believe it. Now, we can prove the identity. Since the system is in equilibrium state, the number of mosquitoes in the bounded region is not changed in any time. Therefore, $\Delta m = \Delta N$.

By so called dimension analysis, we have three freedoms to choose the scale in equation (1) (a) We can take the time scale so that

there are N mosquitoes walk into and walk out the bounded region in a unit time, that is, $N = K\rho cA \cdot 1$. (b) We can choose the speed of random walker is 1, that is, $c = 1$ in random walk. (c) By the definition of density, $\rho = n/V$ the quantity $E(l)A/V$ is dimensionless. We can choose the unit of the length so that $E(l) = 1$ and hence $E(l) = c$. Therefore, equation (1) becomes $1 = KAE(l)/V$. Then we have, $E(l) = V/(KA)$. We have complete the proof of identity (2)

By carefully analyzing, we find two properties of random walk model. One is the property of homogenous, that is, the properties, such as the locations and the associated probability, of the points are translation invariance. The other is the properties of isotropic, that is, the properties, such as the direction of next step and the associated probability, is rotation invariance. Therefore, we find the expectation of the length of random paths is the same as the expectation of the length of random chords. In order to demonstrate the fact, we use two dimensional model to show the result. In Figure 1, there is one random path, from point A to point B then from point B to point C . In Figure 2, there is one random path, from point D to point B then from point B to point E . The events of these two figures are equally likely. In the Figure 3, there are two random paths, one is the same as in Figure 1 and the other is the same as in Figure 2. In the Figure 4, there are two random paths, one is the path from point A to point B then from point B to point E . and the other is the path from point D to point B then from point B to point C . Using these facts, these two cases of events and their associated probabilities, events in Figure 3 and the events in Figure 4, must be equivalent to each other. These arguments support that the expectation of random path (Figure 3) is the same as that of random chords (Figure 4).

The Theoretical Values of K

Now, we are going to find the expectation of random chord. Without losing the generality, we solve the two dimensional problems instead of n -dimensional problem. Though there are so called the Bertrand paradox in random chords problem, we are able solve paradox since the results computer simulation will show the correct answer of this paradox. This verifying work could not be done in that time.

We consider a circle with radius r . The set of all chords is well defined. Therefore, we can find the expectation (average) of the lengths of the set of chords. Since the chords are uniformly and randomly distributed in the circle. we can find some subset of these chords by defining a equivalent relation. The parallel relation between chords is a equivalent relation on the set of that chords. The equivalent relation partitions this set into disjoint equivalent classes. Clearly, the expectations of these equivalent classes must be the same one. In the case of finite set, if the mean of all disjoint subsets is the same one, then the set has the same as that of these subsets. Therefore, we will compute the expectation of the random chords in one of these equivalent classes. In figure 5, the measure of the distribution is $dy/(2r)$. Therefore, we have expectation of random chords, $E(C)$,

$$E(C) = \int_{-r}^r 2xdy/(2r). \quad (3)$$

$$E(C) = \pi r^2/(2r). \quad (3)$$

We interpretate equation (3) as $E(C) = V_n/V_{n-1}$ since the diameter is the normal section of the circle. This interpretation can be generalize to higher dimension cases.

From identity (2), we have

$$K = V_{n-1}/A_n, \quad (4)$$

where V_{n-1} is the volume of S^{n-1} the normal section of S^n and A_n is the boundary area of S^n . Here, we mean that the sphere contains all its interior points. If it is necessary, we will use the term the sphere surface which is S^n to be used in usual sense of mathematician. From equation (4), we are able to calculate the theoretical values of all K .

Discovery of some sequence

The paper, by H. Chalkley, J. Confield and H. Park, stresses how to estimate the ratio of volume and area. So do their followers'. But our interest is to find the values of K . It can be shown that the values of K depend on the dimension the space and that the values of K are not dependent on the shape or the size of the bounded regions. The ratio V/A of S^n is well known. If we can find the expectation of random chords, then we are able to get the theoretical values of K .

From equation (1), We can estimate the value of by computer simulation while the theoretical value of K can be obtained by equation $K = A/(VE(l))$. The values of are $K_1 = 1/2$, $K_2 = 1/\pi$, $K_3 = 1/4$, $K_4 = 2/(3\pi)$, \dots , $K_7 = 5/32$, \dots , $K_{10} = 128/(315\pi)$, \dots .

Our approach is to study the related topics rather than to solve this problem separately or independently. In order to convince the readers that the results we obtain are correct and can be verified by computer simulation, the computer programs have been run on PC, DEC VAX 9000, and IBM SP2 for more than 5 years. It is computer that generates a phenomenon which we have never found. From this phenomenon, we try to find some mathematical model to fit it. It is quite natural to generalize the results to Riemannian manifold. And then the sequence $1/2, 1/\pi, 1/4, 2/(3\pi), \dots, 5/32, \dots, 128/(315\pi), \dots$, is discovered. So far, we have found that K_d 's depend on the dimension of the space or manifold only.

It takes time to judge how important the identity, $E(l) = A/KV$, is, because there are many branches of mathematics such as geometry, analysis and probability or measure theory, concerning with this equation. We think that this is a interesting problem.

We have done the computer simulation of random walk on Riemannian manifold, the surface of the sphere S^n , $n \geq 2$. Almost the same result is obtained. We will organize another paper to show how to design the computer program

for simulating the random walk on the surface of sphere S^n . We shall simply use Riemannian manifold S^n in usual sense. In S^2 , we are shocked by a simple example. Let the boundary region be defined on the north pole of the earth S^2 . The region is a circle of which the center is the north pole and the radius R , $R \leq \pi r/2$, since $R \geq \pi r/2$, the region becomes a circle on the south pole. The chord of the circle is defined as the segment of a geodesic that contains at least one interior point of the circle, the bounded region. Of course, the two end points must be on the circle. Let us use the spherical coordinate system, ρ , θ and ϕ . Here ρ is constant since we are studying the problem on S^2 . For any θ , $0 < \theta \leq \pi/2$, there is a circle defined on the north pole. The value of V is $\int_0^\theta 2\pi r \sin \theta' r d\theta'$, that is, $2\pi r^2(1 - \cos \theta)$. The value of A is $2\pi r \sin \theta$, The value of K is $1/\pi$. Therefore, the value of $E(l)$ or $E(C)$ is $\pi r(1 - \cos \theta)/\sin \theta$, $0 < \theta \leq \pi/2$. The circle is the north hemisphere of the earth when $\theta = \pi/2$. Then the value of V is $2\pi r^2$, the value of A is $2\pi r$ and the value of $E(l)$ or $E(C)$ is πr . We find this is a circle of which all chords have the same length, πr . Since we were confused by this circle, we have spent more than one year's time to reinvestigate our formulation. Finally, we found the result is correct.

References

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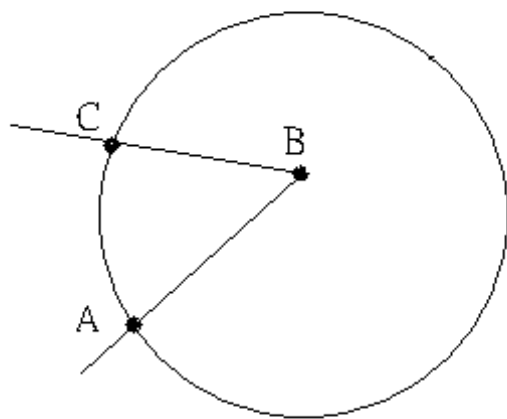


Figure 1: In this figure. There is a mosquito that gets into the region from boundary point A. Then the mosquito makes a turn at point B and goes out the region through the boundary point C.

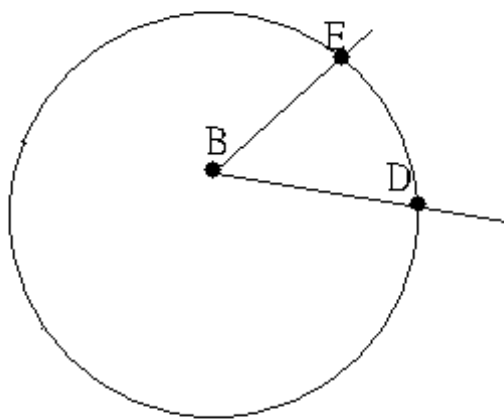


Figure 2: In this figure. There is a mosquito that gets into the region from boundary point D. Then the mosquito makes a turn at point B and goes out the region through the boundary point E.

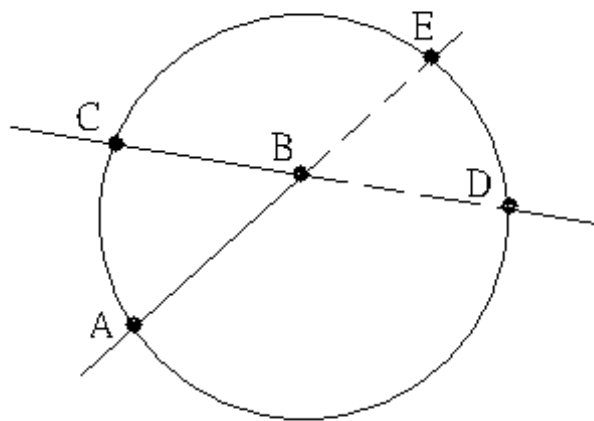


Figure 3: In this figure. There are two mosquitoes that get into the region. one of them gets into the region from boundary point A. Then the mosquito makes a turn at point B and goes out the region through the boundary point C. The other gets into the region from boundary point D. Then the mosquito makes a turn at point B and goes out the region through the boundary point E.

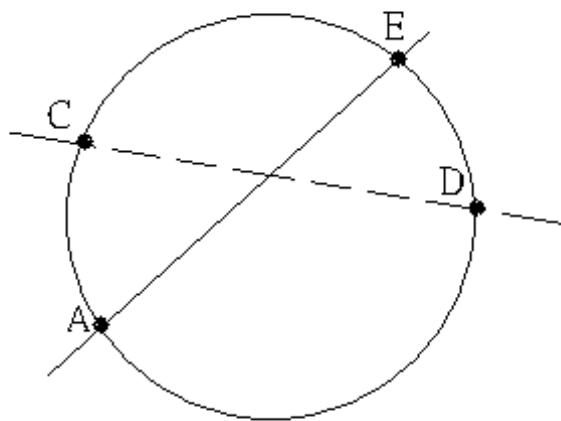


Figure 4: In the figure, there are two random chords, AE and DC in the circle. Clearly, The mean the lengths of these two chords is the same as that of the two random paths in Figure. 3.

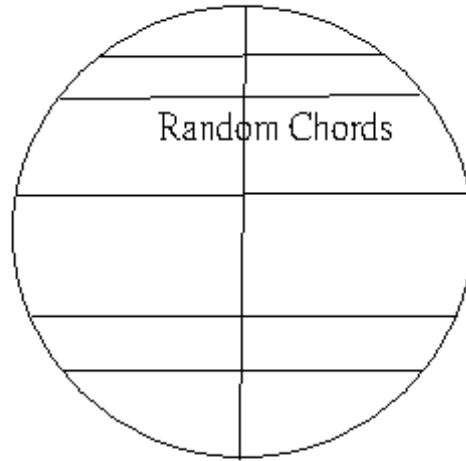


Figure 5: In the figure, some random chords of a class are shown. The diameter that is perpendicular to the random chords is called the normal section of the circle.

the table of data

dimension 1	dimension 2	dimension 3	dimension 4	dimension 5
0.50065	0.313622	0.248796	0.216937	0.188525
0.4994	0.319567	0.262792	0.211288	0.18821
0.58505	0.313033	0.251967	0.213581	0.197106
0.49935	0.319322	0.257308	0.232106	0.185656
0.49895	0.323244	0.252587	0.214341	0.18923
0.53095	0.316656	0.245117	0.211194	0.19041
0.55985	0.346344	0.254642	0.209172	0.195942
0.50035	0.3259	0.251117	0.21765	0.189067
0.4995	0.314611	0.255404	0.217041	0.193394
0.5001	0.314278	0.249083	0.209794	0.188805
0.4988	0.324444	0.250887	0.211703	0.193387
0.49885	0.321067	0.255754	0.206331	0.18697
0.54215	0.3226	0.254733	0.220672	0.190093
0.4994	0.320378	0.263854	0.212503	0.195502
0.50035	0.313078	0.253188	0.207934	0.188976
0.49935	0.321233	0.254737	0.214747	0.196928
0.50005	0.3178	0.247129	0.210894	0.190982
0.5003	0.3166	0.254737	0.215356	0.186613
0.50065	0.349744	0.252467	0.214447	0.195952
0.4988	0.314167	0.251663	0.212528	0.18815
1/2	1/ π	1/4	2/(3 π)	3/16
dimension 6	dimension 7	dimension 8	dimension 9	dimension 10
0.171587	0.15727	0.150828	0.141217	0.13023
0.170157	0.156205	0.146288	0.140869	0.131315
0.170313	0.157762	0.145272	0.136662	0.126905
0.172759	0.158712	0.142739	0.137612	0.130235
0.174857	0.155061	0.144548	0.14564	0.13146
0.178097	0.155077	0.143702	0.140506	0.130815
0.170351	0.15632	0.150119	0.134872	0.12698
0.169589	0.152943	0.147578	0.137501	0.131175
0.173221	0.154071	0.145909	0.136983	0.125865
0.176309	0.161386	0.143639	0.13556	0.129785
0.170673	0.15788	0.146711	0.133069	0.132895
0.167793	0.157454	0.143225	0.136452	0.12908
0.169161	0.160764	0.143214	0.140338	0.13793
0.170093	0.155816	0.144177	0.141294	0.131325
0.175531	0.153232	0.145216	0.135168	0.12902
0.169005	0.158348	0.14707	0.13701	0.12985
0.170316	0.153707	0.14688	0.138528	0.13025
0.17744	0.158312	0.146405	0.134812	0.138405
0.17088	0.160957	0.146869	0.13817	0.131965
0.169431	0.157529	0.147278	0.135202	0.1304
8/(15 π)	5/32	16/(35 π)	35/256	128/(315 π)

The computer program

```
program sphere
  ! This main program. id and k are the control variable for
  ! the dimension of the sace. m is the number
  ! of partcles. n is the steps of random walk.
dimension x(600100,10),dd(600100,10),dx(10),tow(10),tm(10),tn(10)
open(2,file="D:\for\sp41.dat",status="unknown")
rewind(2)
write(2,*)"t=0.01 pro=sp n=1000 4th aprali"
pi=3.1415926
r=1
t=0.01
do id=1,10
  ii=id
  call brn(ii,a,ad)
  nn=a*1.15**id
  n=1000*nn
  m=1000*nn
  do k=1,10
    s=0
    ss=0
    ! The m number of particles is uniformly distributed.
    ! The partcles is generated uniforly in the general interval.
    ! The method of rejction is used to obtain the desired results.
    do i=1,m
      100 rs=0
      do j=1,id
        call random_number(xZ)
        x(i,j)=2*xZ-1
        rs=x(i,j)**2+rs
      end do
      r1=sqrt(rs)
      if (r1 .GE. r)go to 100
    end do
    do ll=1,n
      ss1=0
      ! The directions of next step are generated.
      do i=1,m
        call rt(dx,id)
        do j=1,id
          dd(i,j)=dx(j)
        end do
      end do
      ! The postions of m partcles are updated.
      ! Of course, the counters s and ss1 are used to
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```

! count the number of particles that hit the boundary.
! The technique for using two counter s and ss1 insure
! the sum is correct. That is, sss1=ss1+1 is not correct
! counter, when the content is very large.
do i=1,m
do j=1,id
dx(j)=dd(i,j)
tow(j)=x(i,j)
x(i,j)=x(i,j)+t*dx(j)
tn(j)=x(i,j)
end do
rs=0
do j=1,id
rs=x(i,j)**2+rs
end do
r1=sqrt(rs)
if(r1 .LT. r) go to 300
call tt(tow,tn,r,dx,id,t)
do j=1,id
x(i,j)=tn(j)
dd(i,j)=dx(j)
end do
s=s+1
ss1=ss1+1
300 end do
ss=ss+ss1
ss1=0
end do
! The theoretical value of K, ty , is computed.
ii=id
call voa(ii,a,ad)
tx=n
sm=m
b=a*id
tk=ss/(sm*b*t)*a
ty=ad/b
! The computer simulation of K is tk.
write(2,*)tk,ty
write(*,*)tk,ty
end do
end do
close(2)
stop
end
subroutine tt(tow,tn,r,dx,id,t)
! In this subprogram, the new position is determined

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      ! when the particle collides the boundary of sphere.
      ! The elastic reflection is computed. A simple property is used.
      ! That is, any position, including on the boundary, of the particle is vector
which is
      ! perpendicular to the (tangent of )boundary
dimension tow(10),tn(10),dx(10),f(10),g(10),h(10)
ap=0
pl=0
do j=1,id
ap=ap+dx(j)*tow(j)
pl=pl+tow(j)**2
end do
tp=-ap+sqrt(ap**2+1-pl)
pr=0
pr1=0
do j=1,id
f(j)=tow(j)+tp*dx(j)
pr=pr+(tn(j)-f(j))*f(j)
pr1=pr1+dx(j)*f(j)
end do
dl=0
do j=1,id
tn(j)=tn(j)-2*pr*f(j)
dx(j)=dx(j)-2*pr1*dx(j)
dl=dl+dx(j)**2
end do
dl=1.0/sqrt(dl)
do j=1,id
dx(j)=dx(j)*dl
end do
return
end
subroutine rt(dx,id)
      ! In this subprogram a random unit vector is generated for
      ! direction of next step random walking.
dimension dx(10)
      100 tl=0
do j=1,id
call random_number(xz)
dx(j)=2*xz-1
tl=tl+dx(j)**2
end do
if(tl .EQ. 0) go to 100
tl=sqrt(tl)
if(tl .GT. 1) go to 100
tl=1.0/tl

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do j=1,id
dx(j)=dx(j)*tl
end do
return
end
subroutine voa(ii,a,ad)
    ! In the subprogram, both the volumes of unit sphere and the normal section
    ! section of the unit sphere are computed. For example, if input ii=3,
    ! then output a is the volume of three dimensional unit sphere and
    ! the other output ad is the area of a unit two dimensional sphere, the disk.
ii=ii-1
call brn(ii,a,ad)
ad=a
ii=ii+1
call brn(ii,a,ad)
return
end
subroutine brn(ii,a,ad)
    ! In this subprogram, the volume of unit sphere is computed
    ! The parameter ii is the dimension, the parameter a is ouput and
    ! the parameter ad is a redundancy
in=ii/2
in=ii-2*in+1
if (in .EQ. 2)go to 400
f=1
ls=ii/2
do j= 1,ls
f=f*j
end do
a=3.1415926**ls/f
go to 500
400 f=1
do j=1,ii
f=f*j
end do
l1=(ii-1)/2
f1=1
do j=1,l1
f1=f1*j
end do
a=3.1415926**l1*2**ii*f1/f
500 return
end

```